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Abstract

In this paper we consider the case of standard fixed tree games, where each vertex unequal to the root is inhabited by exactly one player. We present two weighted allocation rules, the weighted down-home allocation and the weighted neighbour-home allocation, both inspired by the *painting story* in Maschler *et al.* (1995) . We show, in a constructive way, that the core equals both the set of weighted down-home allocations and the set of weighted neighbour allocations. Since every weighted down-home allocation specifies a weighted Shapley value (Kalai and Samet (1988)) in a natural way, and vice versa, our results provide an alternative proof, which is constructive, of the fact that the core of a standard fixed tree game equals the set of weighted Shapley values. The class of weighted neighbour allocations is a generalization of the nucleolus, in the sense that the latter is in this class as the special member where players have all equal weights.

JEL Classification: C71

Keywords: Cooperative game theory, tree games, core, weighted Shapley value, nucleolus

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1 Introduction

We consider cost sharing problems arising from *standard fixed tree enterprises*. There is a fixed and finite set of agents who are connected through a fixed tree network to a special vertex called *root*. We seek to allocate the cost of this tree corresponding to the maintenance of the different connections. Many real-life situations can be modelled to fit in this general setting. For instance, consider the problem of allocating the maintenance cost of an irrigation network or a cablevision network, setting airport taxes for planes or setting dredging fees for ships. In a natural way each standard fixed tree problem gives rise to a *standard fixed tree game*, which relates each coalition of agents/players to the minimal expenses for maintaining the connections of all its members to the root. This makes it possible to investigate this type of problems with techniques from cooperative game theory. The same problem is studied in Megiddo (1978) , Koster *et al.* (1998) whereas Granot *et al.* (1996), Maschler and Granot (1998) and Maschler *et al.* (1995) study a generalization, where more than one player is allowed to occupy each vertex. A special case, where the underlying structure of the game is a chain, is also known as the airport problem and is considered by Littlechild (1974), Littlechild and Owen (1977), Littlechild and Thompson (1977), Dubey (1982), Potters and Sudhölter (1999) , and Aadland and Kolpin (1998). In section 3 we are concerned with the core of the standard fixed tree game in section 3; essentially it is based on Koster *et al.* (1998) . Inspired by the *painting story* presented by Maschler *et al.* (1995) we introduce, the *weighted down-home allocation* in section 4. Accordingly, each player is allocated a share, corresponding to his relative weight, of the cost of each arc along the path from the (local) root to his home. We show, by explicitly characterizing the corresponding weight system, that each core element can be obtained as a weighted down-home allocation. Especially, the core element as determined by the Shapley value corresponds to the weighted down-home allocation with equal weights to all players. Moreover, each weighted down-home allocation is equivalent to a weighted Shapley value, and therefore our results provide an alternative proof of the result in Monderer *et al.* (1992), where it is shown that the core of a concave game (it is well known that fixed tree games are concave) equals the set of weighted Shapley values. In section 5 we introduce the *weighted neighbour-home allocation*, a generalization of the scheme in Maschler *et al.* (1995) for computing the nucleolus, and show that the set of weighted neighbour-home allocations equals the core. The weighted neighbour-home allocation is equal to the nucleolus in the special case where all players are given equal weight. But first, in section 2, we formally define the standard fixed tree problem and its corresponding game, and introduce the necessary notation.

2 The fixed tree maintenance problem: the model and its game

In this paper we consider a *fixed tree maintenance problem* $\mathcal{G} := (G, c, N)$. Here $G = (V, E)$ is a tree, i.e. a directed connected graph without cycles, with vertex set V and arc set E. The set V contains a vertex which has a special meaning. We denote this vertex by r and refer to it as the *root*. The function $c : E \to \mathbb{R}_+$, called *cost function*, associates with each arc e a cost $c(e)$. It can be interpreted as the cost to maintain e. At each vertex there is exactly one player, the finite set of all players is denoted by $N = \{1, \ldots, n\}$ for some natural number n. The objective of the players is to maintain sufficiently many arcs such that by the corresponding network each finds himself connected to the root. We address the problem of allocating the total costs of the network among the players.

We assume, for simplicity, that the root is not occupied and that only one arc is incident with the root. Then $\mathcal{G}$ is referred to as simply a *maintenance problem.* In the sequel we identify vertices with players ($V = N \cup \{r\}$). For any subgraph G' of G, we will let $E(G')$ and $V(G')$ denote the corresponding arc set and vertex set, respectively. Sometimes we will also denote the player set corresponding to G' by $N(G') \subseteq V(G')$. For each vertex $i \in N$ there is a unique path P_i from the root to vertex i. If $V(P_i)$ consist of the vertexes $j_0 = r, j_1, \ldots, j_q = i$, then j_{q-1} is called the *predecessor* $\pi(i)$ of vertex i. We put $N(P_i) := V(P_i) \setminus \{r\}$. We denote by e_i the arc $(\pi(i), i)$, and we will sometimes write $c_i := c(e_i)$. The *precedence relation* $(V, \preceq)$ on the set of vertices and/or players is defined by $i \preceq j$ if and only if $i \in V(P_j)$. Analogously we define the precedence relation $(E, \preceq)$ on the arcs. In this way, the arcs are considered to be directed away from the root. A *trunk* of $G = (V, E)$ is a set of vertices $T \subseteq N$, which is closed under the precedence relation defined above, i.e. if $i \in T$ and $j \preceq i$, then $j \in T$. Also, let the *followers* of a vertex i be denoted by $F(i) := \{j \in N : i \preceq j\}$. A vertex i is called a *leaf* if $F(i) = \{i\}$. If $e = (i, j)$, then by B_e we denote the *branch* at i in the direction of j, i.e. the subgraph defined by $V(B_e) := \{i\} \cup F(j)$, $N(B_e) := F(j)$ and $E(B_e) := \{(k, \ell) \in E : k, \ell \in V(B_e)\}$.

In this paper we study the maintenance problem in the setting of cooperative game theory, by associating each maintenance problem with an appropriate cooperative *cost game*. Here by a *cost game* is meant an ordered pair (N, g) consisting of a finite set N of *players* and $g : 2^N \to \mathbb{R}$ is the *characteristic function* relating each coalition of players $S \subseteq N$ to a real number $g(S)$ that is interpreted as the total cost of serving the collective S. Moreover, it is assumed that $g(\emptyset) = 0$, i.e. serving nobody can be done at no cost. The set of all cost games is denoted Γ, and the restriction to cost games with player set N is denoted Γ^N. Each maintenance problem $\mathcal{G} = (G, c, N)$ is naturally related to the cost game $(N, c_{\mathcal{G}}) \in \Gamma$, where the cost $c_{\mathcal{G}}(S)$ of each coalition S is defined as the minimal cost needed to maintain all connections of the members of S to the root via a connected subgraph of (V, E), i.e.

$$c_{\mathcal{G}}(S) = \sum_{i \in T_S} c(e_i) \text{ for all } \quad \emptyset \neq S \subseteq N, \tag{2.1}$$

where $T_S = \{i \in N : \exists\, j \in S \text{ with } i \preceq j\}$, and $c_{\mathcal{G}}(\emptyset) = 0$. T_S is the smallest trunk containing S.

In order to prove some of our results, we will need to represent our cost game using the basis $\{(N, u^*_S)\}_{S \subseteq N}$ of dual unanimity games. For $S \subseteq N$, the game (N, u^*_S) is defined by

$$u^*_S(T) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $T \subseteq N$. It is known (see Koster *et al.* (1998)), that if $\mathcal{G} = (G, c, N)$ is a maintenance problem, then the associated cost game $(N, c_{\mathcal{G}})$ can be represented as

$$c_{\mathcal{G}} = \sum_{e \in E} c(e) u^*_{N(B_e)}. \tag{2.2}$$

3 The core of a maintenance game

Given a maintenance problem $\mathcal{G} = (G, c, N)$, we want to address the problem of dividing the total maintenance cost $c_{\mathcal{G}}(N)$ among the players in N. A particular division is represented by a vector $x \in \mathbb{R}^N$, such that x_i stands for the contribution of player $i \in N$ and $\sum_{j \in N} x_j = c_{\mathcal{G}}(N)$. Such an element x will be referred to as a *vector of cost shares*. We will be concerned with those vectors of cost shares that are stable in the sense that no coalition of players in N has an incentive to split off. In this respect, a vector of cost shares $x \in \mathbb{R}^N$ is a *core element* of $(N, g) \in \Gamma$ if for all $S \subseteq N$ it holds that $\sum_{i \in S} x_i \leq g(S)$. The set of all core elements of (N, g) is called the *core* of (N, g), and is denoted $C(g)$. It is a well-known result (Shapley (1971)) that $C(g) \neq \emptyset$ if g is *concave*, i.e. if for all $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$, $g(S \cup \{i\}) - g(S) \geq g(T \cup \{i\}) - g(T)$. In particular maintenance games have a nonempty core, since they are concave. Below we will focus on characterizations of the core of maintenance games.

First we introduce some additional notation. For $y \in \mathbb{R}^N$ and $S \subseteq N$, $y^S \in \mathbb{R}^S$ denotes the restriction of y to S. It will be convenient to define $y(S) := \sum_{i \in S} y_i$. We will use the column representation of a vector. The transpose of a vector x is denoted x^T.

Given some vector of cost shares $x \in \mathbb{R}^N$, we define the *overflow* over the arc e as the amount that the members of $N(B_e)$, i.e. the inhabitants of the branch B_e, pay in excess of the cost of the arcs of B_e, i.e.

$$O_e(x) := \sum_{i \in N(B_e)} x_i - \sum_{f \in E(B_e)} c(f) = \sum_{i \in N(B_e)} (x_i - c_i).$$

If $e = (i, j)$, we will sometimes write $O_j(x)$ instead of $O_e(x)$, and it is easily seen that

$$O_j(x) = \sum_{\ell \in F(j)} (x_\ell - c_\ell) = (x_j - c_j) + \sum_{\ell \in \pi^{-1}(j)} O_\ell(x). \tag{3.1}$$

Characterizations of the core of the game $(N, c_{\mathcal{G}})$ are found in Koster *et al.* (1998) and Granot and Maschler (1998). The following proposition summarizes these results and adds a characterization of the core in terms of the overflows.

Proposition 3.1 *Let $x \in \mathbb{R}^N$. Then the following statements are equivalent:*

(i) $x \in C(c_{\mathcal{G}})$

(ii) $x(N) = c_{\mathcal{G}}(N)$, $x \geq 0$, *and* $x(T) \leq c_{\mathcal{G}}(T)$ *for every trunk* T.

(iii) $x(N) = c_{\mathcal{G}}(N)$, $x \geq 0$, *and* $O_e(x) \geq 0$ *for all* $e \in E$.

(iv) *There exist* $y^e \in \Delta(N(B_e))$ *for all* $e \in E$, *such that*

$$x_i = \sum_{e \in E(P_i)} y_i^e c(e) \qquad\qquad \textit{for all } i \in N.$$

Proof. These results essentially appear as Propositions 3.1 ($(i) \Leftrightarrow (ii)$), 3.2 ($(ii) \Leftrightarrow (iii)$), and 3.3 ($(i) \Leftrightarrow (iv)$) in Koster *et al.* (1998). □

Granot and Maschler (1998) shows that in characterizing the core of a standard fixed tree game, in relation to the above statement (ii), one needs only to consider the trunks T with one *outgoing arc*, i.e. those trunks with the property that $\{j \in N \backslash T, \mid \pi(j) \in T\}$ consists of precisely one element.

Definition 3.2 A *pseudo subtree* of a tree $G = (V, E)$ is a connected subgraph $G' = (V', E')$ such that there exists an $r' \in V(G')$ such that

(i) r' is the minimal element in $V(G')$ with respect to $\preceq$,

(ii) there is exactly one vertex in $V(G')$ that has r' as predecessor.

Definition 3.3 A pseudo subtree $G' = (V(G'), E(G'))$ of G rooted at r' defines a *restricted maintenance problem* $\mathcal{G}' = (G', c', N')$ where c' is the restriction of c to E', and $N' := V(G') \setminus \{r'\}$.

Definition 3.4 Let $\mathcal{T} = (G^1, \ldots, G^p)$ be an ordered collection of pseudo subtrees of G. Then $\mathcal{T}$ is said to be a *partition of G into pseudo subtrees* if and only if

(i) for all $k = 1, \ldots, p$, there exists $r_k \in V(G^k)$ such that G^k is the pseudo subtree of G rooted at r_k,

(ii) $(N(G^1), \ldots, N(G^p))$ is a partition of N.

Given an allocation vector x, let $E(x) := \{e \in E : O_e(x) > 0\}$. The graph $(V, E(x))$ contains p connected subgraphs, where $1 \leq p \leq n$. For each of these subgraphs, $1 \leq k \leq p$, we construct a pseudo pseudo subtree G^k with player set $N(G^k)$. Let $r_k \in V \setminus N(G^k)$ be such that $r_k \in V(P_i)$ for every $i \in N(G^k)$, and $r_k = \pi(i)$ for exactly one $i \in N(G^k)$. Let $V(G^k) := N(G^k) \cup \{r_k\}$ and $E(G^k) := \{e = (i,j) : i,j \in V(G^k)\}$. Then $G_k := (V(G^k), E(G^k))$ is a pseudo subtree rooted at r_k, and $\mathcal{T}(x) := (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees. We will refer to $\mathcal{T}(x)$ as the partition of G *induced by* x.

Example 3.5 Consider the maintenance problem $\mathcal{G} = (G, c, N)$ described by figure 3.1, where the arc weights are given by $c(e) := 10$ for all $e \in E$. The allocation $x = (4, 5, 15, 16)$ is a core element, and the corresponding overflows are indicated next to the arcs in the figure. By removing all the arcs with zero overflows, we obtain the partition of G into the pseudo subtrees G^1 and G^2, where $N(G^1) = \{1, 4\}$, $N(G^2) = \{2, 3\}$, $r_1 = r$, and $r_2 = 1$. $\triangleleft$

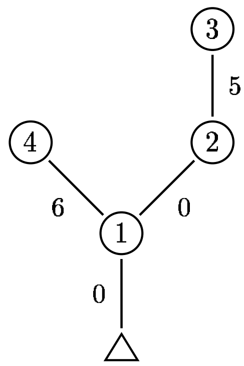

Figure 3.1: The tree of Example 3.5.

For any $i \in N$, let $1 \leq k(i) \leq p$ be such that $i \in N(G^{k(i)})$. Let, for any $i \in N$, $\tilde{F}(i) := F(i) \cap V(G^{k(i)})$. For $1 \leq k \leq p$ and $e \in E(G^k)$, let $\tilde{B}_e$ be defined such that $V(\tilde{B}_e) := V(B_e) \cap V(G^k)$, $N(\tilde{B}_e) := N(B_e) \cap N(G^k)$, and $E(\tilde{B}_e) := E(B_e) \cap E(G^k)$. In an analogous manner, for $1 \leq k \leq p$ and $i \in V(G^k)$, define $\tilde{P}_i$. We will write $\tilde{O}_e(x) :=$

$\sum_{i \in N(\tilde{B}_e)}(x_i - c_i)$.

Proposition 3.6 *Let $\mathcal{T} = (G^1, \ldots, G^p)$ be a partition of G into pseudo subtrees, and let $\mathcal{G}^1, \ldots, \mathcal{G}^p$ be the corresponding induced maintenance problems.*

(i) The Cartesian product of the cores of the cost games $(N(G^1), c_{\mathcal{G}^1}), \ldots, (N(G^p), c_{\mathcal{G}^p})$ is included in the core of $(N, c_{\mathcal{G}})$, i.e. $\prod_{k=1}^{p} C(c_{\mathcal{G}^k}) \subseteq C(c_{\mathcal{G}})$.

(ii) Let $x \in C(c_{\mathcal{G}})$, and suppose that $\mathcal{T}$ is such that $\mathcal{T} = \mathcal{T}(x)$. Then x is contained in the Cartesian product of the cores of the games $c_{\mathcal{G}^1}, c_{\mathcal{G}^2}, \ldots, c_{\mathcal{G}^{p-1}}$ and $c_{\mathcal{G}^p}$, i.e.

$$x \in \prod_{k=1}^{p} C(c_{\mathcal{G}^k}).$$

Proof. These results appear as Proposition 3.4 (i) and (ii) respectively in Koster *et al.* (1998). Here we will give an alternative prove of (ii), thereby using the core characterization in Proposition 3.1(iii). Let $1 \leq k \leq p$. Because $x \in C(c_{\mathcal{G}})$ is a vector of cost shares with respect to the game $c_{\mathcal{G}}$, and since $\mathcal{T}$ has been constructed by removing only arcs with zero overflows, it is clear that $x^{N(G^k)}$ is a vector of cost shares with respect to the game $c_{\mathcal{G}^k}$. Also, $x^{N(G^k)} \geq 0$ follows from $x \in C(c_{\mathcal{G}})$ and Proposition 3.1. We will complete the proof by showing that $\tilde{O}_i(x) = O_i(x) \geq 0$ for all $i \in N(G^k)$, where the inequality follows from $x \in C(c_{\mathcal{G}})$ and Proposition 3.1(iii). Note that, by (3.1) and the construction of $\mathcal{T}$, $\tilde{O}_i(x) = x_i - c_i = O_i(x)$ for any $i \in N(G^k)$ such that i is a leaf in G^k, since i must either be a leaf in G, or we must have $O_j(x) = 0$ for every $j \in \pi^{-1}(i)$. Then, for every $i \in N(G^k)$ such that i is not a leaf in G^k, $\tilde{O}_i(x) = (x_i - c_i) + \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \tilde{O}_j(x) = (x_i - c_i) + \sum_{j \in \pi^{-1}(i)} O_j(x) = O_i(x)$. □

4 The core and the set of weighted down-home allocations

A well-known single-valued solution concept for cooperative cost games is the Shapley value (Shapley (1953)). In general, despite its attractiveness as a solution concept, the computational complexity of finding the Shapley value can be troublesome. However, nice expressions are known for airport games (cf. Littlechild and Thompson (1977), Dubey (1982), Koster *et al.* (1998)). Koster *et al.* (1998) show a similar result for maintenance games. Roughly, the Shapley value of a maintenance game is obtained through a dynamical process of uniformly distributing the costs of the arcs from the root in the direction of leafs. In this section we will show this procedure and that by a simple adaptation of this algorithmic approach we obtain the class of weighted Shapley values (Kalai and Samet (1988)). First we will develop the dynamical approach that specifies a *weighted down-home allocation.* Then afterwards we conclude that it represents no more than a weighted Shapley value. It is the result of Monderer *et al.* (1992) that the weighted Shapley values cover the core of a concave game. Opposed to their result we show in a *constructive* way that each core element of the maintenance game corresponds to a weighted Shapley value.

Definition 4.1 Let $\mathcal{G} = (G, c, N)$ be a maintenance problem. Let $\mathcal{T} = (G^1, \ldots, G^p)$ be a partition of G into pseudo subtrees, and let $w \in \mathbb{R}^N_+$ be such that $w(\tilde{F}(i)) > 0$ for all $i \in N$ such that $c(e_i) > 0$. Then $(\mathcal{T}, w)$ is called a *weight system for* $\mathcal{G}$. The set of all weight systems for $\mathcal{G}$ is denoted by $\mathcal{B}(\mathcal{G})$.

Consider a maintenance problem $\mathcal{G} = (G, c, N)$ and some weight system $\beta \in \mathcal{B}(\mathcal{G})$. For each pseudo subtree G^k, interpret the vertices in $N(G^k)$ as the homes of the different players and the arcs in $E(G^k)$ as the roads to the community center (r_k). The cost of a road is expressed as the number of days it takes (for one person) to paint the stripes on the road. The work is done by the players themselves according to the following rules[1]:

(i) Every worker keeps painting as long as the road from the community center to his home has not been completed.

(ii) Every worker does his job on an unfinished segment between the community center and his home.

(iii) Every worker starts painting at the same moment.

(iv) Every worker $i \in N$ paints with velocity w_i.

[1]These rules are inspired by the *painting story* presented in Maschler *et al.* (1995).

(v) Each worker paints as close to the community center as the rules (i)-(iv) permit him to.

We call the resulting allocation the *weighted down-home*[2] *allocation*, and denote it $\delta^{\beta}(\mathcal{G})$. It is given for any player $i \in N$ by

$$\delta_i^{\beta}(\mathcal{G}) = \sum_{e \in E(\tilde{P}_i)} \frac{w_i}{w(N(\tilde{B}_e))} c(e). \tag{4.1}$$

Example 4.2 Consider the example illustrated in Figure 4.1, where $c(e) := 10$ for every $e \in E$. Let $\mathcal{T} := (G^1, G^2)$, where $N(G^1) := \{1,2,3\}$ and $N(G^2) := \{4\}$, and let $w := (1,1,3,1)^T$. For $\beta := (\mathcal{T}, w)$ we have $\delta^{\beta}(\mathcal{G}) = \left(2, 4\frac{1}{2}, 23\frac{1}{2}, 10\right)$. Player 1 only contributes to the cost of arc $(r,1)$, so his total contribution is $10 \cdot \frac{1}{5} = 2$. Player 2 contributes to the cost of arc $(r,1)$ and $(1,2)$, with relative weights of $\frac{1}{5}$ and $\frac{1}{4}$, respectively, so his total contribution is $10 \cdot \frac{9}{20} = 4\frac{1}{2}$. Player 3 contributes at arc $(r,1)$, $(1,2)$, and $(2,3)$, with relative weights of $\frac{3}{5}$, $\frac{3}{4}$, and 1, respectively, hence his total contribution is $10 \cdot \frac{47}{20} = 23\frac{1}{2}$. Player 4 is the only player in his pseudo subtree, and contributes the entire cost of the arc that he uses, i.e. 10. $\triangleleft$

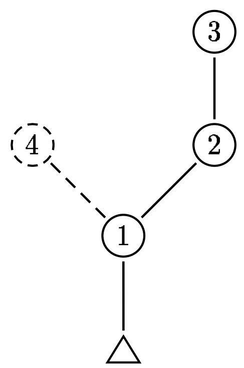

Figure 4.1: Tree of Example 4.2

From Proposition 3.1 it follows that each down-home allocation specifies a core-element. But as we are about to show, the converse also holds. For each core element x there is a

[2] Koster *et al.* (1998) treat the weighted *home-down* allocation, which results by replacing "the community center" in (v) by "his home". The resulting allocation is related to a weighted version of the *constrained egalitarian solution* of Dutta and Ray (1989), see, e.g., Koster (1999) or Hokari (1998).

weight system β such that the corresponding down-home allocation $\delta^{\beta}(\mathcal{G})$ equals x. We will show how such a weight system β is easily calculated for a given $x \in C(c_{\mathcal{G}})$.

First of all the partition of the player set is derived from the partition of $\mathcal{G}$ into pseudo subtrees induced by x; this can be done by considering the overflows in the tree. Next the weights for the players are calculated for each separate subproblem. The idea hinges on the following observations. We assume that the partition into pseudo subtrees of $\mathcal{G}$ with respect to x is trivial, or, equivalently, all the overflows are positive except at the arc that leaves the root.

Without loss of generality we will assume that player 1 is the player directly connected to the root. The cost of the corresponding arc e_1 is covered by the collective of players N. Suppose that x is a down-home allocation. Then our objective is, if at all possible, to find a suitable vector of weights w such that for $\beta = (\{G\}, w) \in \mathcal{B}(\mathcal{G})$ we have $\delta^{\beta}(\mathcal{G}) = x$. First of all, with the interpretation of the weights as painting speeds, the arc e_1 is painted in

$$\frac{c(e_1)}{w(N)} = c(e_1)$$

units of time, if we assume that w is normalized such that $w(N) = 1$. Moreover, each of the painting players is finished with e_1 at the same time. In particular, if player 1 is painting at all (in case $x_1 > 0$) then he is also painting for $c(e_1)$ units of time. On the other hand he must complete x_1 by himself, at speed w_1, so we have the condition

$$\frac{x_1}{w_1} = c(e_1),$$

and thus

$$w_1 = \frac{x_1}{c(e_1)}.$$

Note that $c(e_1) > 0$, since $O_i(x) > 0$ for all $i \in \pi^{-1}(1)$. After having calculated this first weight, we proceed by consecutively assigning weights to each of the players in the sets $\pi^{-1}(1), \pi^{-1}(\pi^{-1}(1)), \ldots,$ until even the leaf players have a weight. Basically we repeat the above type of reasoning. Consider a player $i \notin \pi^{-1}(1)$. Then, according to x, his followers $F(i)$ contribute $O_i(x) > 0$ to the maintenance cost of the path from the root to his predecessor, player $\pi(i)$. Recall again the painting story. The speed at which the collective of players $F(i)$ operates on the path from r to $\pi(i)$ is given by the aggregate of the weights $w(F(i))$. Then the time that the group of players $F(i)$ needs to complete $O_{F(i)}(x)$ is given by

$$\frac{O_i(x)}{w(F(i))}.$$

Similarly, it holds that the followers of $\pi(i)$ contribute $O_{\pi(i)}(x)$ to the path from the root

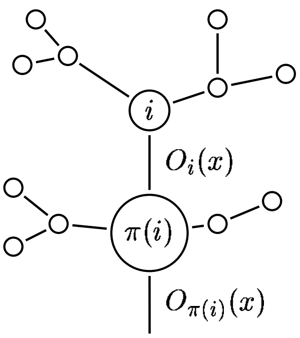

Figure 4.2: A tree network with overflows.

to $\pi(\pi(i))$ *plus* the full cost of maintaining the arc $(\pi(i), \pi(\pi(i)))$. The collective of players $F(\pi(i))$ paints at speed $w(F(\pi(i)))$, which means that the time that it needs to complete their part of the path from the root to $\pi(i)$ equals

$$\frac{O_{\pi(i)} + c(e_{\pi(i)})}{w(F(\pi(i)))}.$$

This expression indicates the time that each of the individuals in $F(\pi(i))$ is working on the path from r to $\pi(i)$, and especially each of the players in $F(i)$. But then we must have the equality

$$\frac{O_i(x)}{w(F(i))} = \frac{O_{\pi(i)}(x) + c(e_{\pi(i)})}{w(F(\pi(i)))}.$$

This determines an iterative procedure for calculating all the weights $w(F(i))$ for each $i \in F(1)\backslash\{1\}$, since

$$w(F(i)) = w(F(\pi(i)))\frac{O_i(x)}{O_{\pi(i)}(x) + c(e_{\pi(i)})}$$

for all $i \in N$, and consequently

$$w_i = w(F(i)) - \sum_{j \in \pi^{-1}(i)} w(F(j)).$$

Example 4.3 Consider the network $\mathcal{G}$ depicted in Figure 4.3. As in earlier examples the maintenance costs of the different arcs are all 10. Check that $x = (5, 13, 12)^T$ is a core element for $c_{\mathcal{G}}$. The numbers at the arcs in Figure 4.3 denote the overflows corresponding to x. Firstly, observe that the partition $\mathcal{T}$ of $\mathcal{G}$ into pseudo subtrees induced by x is trivial. Assume that x is a down-home allocation: there is a vector of weights w with $w_i > 0$ for all $i \in \{1, 2, 3\}$ such that $\delta^\beta(\mathcal{G}) = x$ for $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$. Recall the painting story for the weighted down-home allocation. The players 1,2, and 3 respectively paint at velocities w_1, w_2, and w_3 at e_1 as long as $c(e_1) = 10$ is not completed. Furthermore, the contribution of player 1 and the overflows $O_2(x)$ and $O_3(x)$ determine the parts of $c(e_1)$ that are individually covered by the players 1,2 and 3 respectively. Given the velocities we can compute the time that the players need to finish these parts in three ways, as

$$\frac{x_1}{w_1}, \frac{O_2(x)}{w_2}, \quad \text{or} \quad \frac{O_3(x)}{w_3}.$$

These numbers are equal by the fact that all the players will continue painting on e_1 until it is finished, which implies that the finishing time of the collective of players equals the individual finishing times. Since we are completely informed about the individual

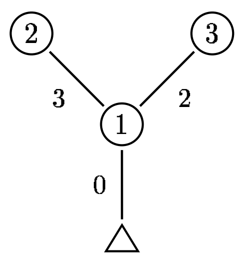

Figure 4.3: The tree of Example 4.3.

contribution of player 1 and the overflows corresponding to each branch emanating from the node of player 1, we must therefore have

$$\frac{5}{w_1} = \frac{3}{w_2} = \frac{2}{w_3},$$

and thus $w = (w_1, \frac{3}{5}w_1, \frac{2}{5}w_1)^T$. If w is required to be a vector in the unit simplex, we get $w_1 = \frac{1}{2}$ and $w = (\frac{1}{2}, \frac{3}{10}, \frac{2}{10})^T$. The reader may verify that indeed $\delta^\beta(\mathcal{G}) = x$ for $\beta = (\{G\}, w)$. $\triangleleft$

Example 4.4 Consider the network as in Figure 4.4. All arcs have equal maintenance cost 10. Consider the core element $x = (4, 12, 12, 12)^T$ of the corresponding 4-player maintenance game. The overflows corresponding to x are the numbers to the arcs. The

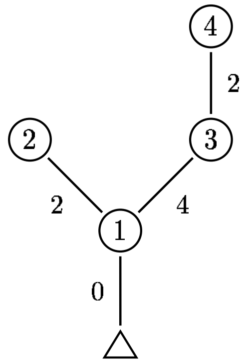

Figure 4.4: The tree of Example 4.4.

partition into pseudo subtrees by x is trivial. Assume that x is a down-home allocation, i.e. there is a vector $w \in \mathbb{R}^4$ with all positive coordinates such that for $\beta = (\{G\}, w)$ we have $\delta^\beta(\mathcal{G}) = x$. We will see that similar reasoning as in the above example 4.3 leads to conditions that determine w. Basically, the only difference with the situation in example 4.3 is that it is not directly clear what are the individual contributions of the players 3 and 4 at e_1. We are only able to monitor their aggregate efforts by means of $O_3(x)$. The same considerations as in the above example lead to the conclusion that players 1,2, and the collective of players 3 and 4 respectively finishes in $\frac{x_1}{w_1}$, $\frac{O_2(x)}{w_2}$ and $\frac{O_3(x)}{w_3+w_4}$ time units respectively. Since these numbers are all equal we have

$$\frac{4}{w_1} = \frac{2}{w_2} = \frac{4}{w_3 + w_4}.$$

Therefore, at this stage we are able to express w_2 and $w_3 + w_4$ in terms of w_1, i.e. $w_2 = \frac{1}{2}w_1$, $w_3 + w_4 = w_1$. If we require $w(N) = 1$, we get $w_1 = \frac{2}{5}$. This means that we only have to consider w_3 and w_4 since $w_2 = \frac{1}{2}w_1 = \frac{1}{5}$. Consider the path from the root to vertex 3. The players 3 and 4 reach vertex 3 at the same time. The time they need to complete the path from the root to vertex 3 equals the time for finishing e_1 plus the time necessary for completing e_3, i.e.

$$\frac{O_3(x)}{w_3 + w_4} + \frac{c(e_3)}{w_3 + w_4} = \frac{O_3(x) + c(e_3)}{w_3 + w_4}.$$

At this precise moment player 4 has completed exactly $O_4(x)$. Using the velocity of player 4, w_4, therefore the time that player 4 must spend equals $\frac{O_4(x)}{w_4}$ and thus

$$\frac{O_4(x)}{w_4} = \frac{O_3(x) + c(e_3)}{w_3 + w_4}.$$

This means that

$$\frac{2}{w_4} = \frac{14}{w_3 + w_4} = 35,$$

from which we see that $w_4 = \frac{2}{35}$ and $w_3 = w_1 - w_4 = \frac{2}{5} - \frac{2}{35} = \frac{12}{35}$. Thus $w = \left(\frac{2}{5}, \frac{1}{5}, \frac{12}{35}, \frac{2}{35}\right)^T$.

$\triangleleft$

Now we will formalize the above ideas. For any core allocation x, we define a weight system $\beta \in \mathcal{B}(\mathcal{G})$ such that such that $x = \delta^{\beta}(\mathcal{G})$. First, find the partition $\mathcal{T} = (G^1, \ldots, G^p)$ of G into pseudo subtrees induced by x. Then a weight vector w can be found by first, for all $i \in N$, calculating the sums

$$w(\tilde{F}(i)) = \begin{cases} 1 & \text{if } \pi(i) = r_{k(i)}, \\ \dfrac{\tilde{O}_i(x)}{\tilde{O}_{\pi(i)}(x) + c_{\pi(i)}} w(\tilde{F}(\pi(i))) & \text{else,} \end{cases} \tag{4.2}$$

in a recursive manner, and then the individual weight for a player $i \in N$ is given by

$$w_i = w(\tilde{F}(i)) - \sum_{j \in \tilde{F}(i)} w(\tilde{F}(j)) \tag{4.3}$$

Proposition 4.5 *Let $x \in C(c_{\mathcal{G}})$. There exists $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $x = \delta^{\beta}(\mathcal{G})$, where $\mathcal{T} = \mathcal{T}(x)$, and w satisfies* (4.2) *and* (4.3).

Proof. First we show the existence part. Observe that $\mathcal{T}(x)$ exists and that if, for some $i \in N$, we have $|N(G^{k(i)})| = 1$, then (4.2) and (4.3) imply $w_i = 1$. To prove existence for w, it is therefore sufficient to show that

$$\tilde{O}_i(x) + c_i > 0 \text{for all } i \in N \text{ such that } |N(G^{k(i)})| > 1. \tag{4.4}$$

Since $c_i \geq 0$ for all $i \in N$, and since $\tilde{O}_i(x) > 0$ for all $i \in N$ such that $\pi(i) \neq r_{k(i)}$, the only possible problem arises if $c_i = 0$ for a player i such that $\pi(i) = r_{k(i)}$. Suppose that this is the case. Then, since, by the construction of $\mathcal{T}$, $x^{N(G^{k(i)})}$ is a vector of cost shares with respect to the game $c_{\mathcal{G}^{k(i)}}$, we must have $\tilde{O}_j(x) = 0$ for all $j \in \pi^{-1}(i) \cap \tilde{F}(i)$, contradicting the fact that $\mathcal{T}$ is induced by x.

$\beta \in \mathcal{B}(\mathcal{G})$: Clearly, $\mathcal{T} = (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees. From (4.2), (4.4), and because $\tilde{O}_i(x) > 0$ if $\pi(i) \neq r_{k(i)}$, it follows that

$$w(\tilde{F}(i)) > 0 \qquad \text{for all } i \in N. \tag{4.5}$$

Also, for any $i \in N$, we have from (4.2) and (4.3) that

$$\begin{aligned} w_i &= w(\tilde{F}(i)) \left\{ 1 - \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \frac{\tilde{O}_j(x)}{\tilde{O}_i(x) + c_i} \right\} \\ &= w(\tilde{F}(i)) \frac{\sum_{j \in \tilde{F}(i)} (x_j - c_j) + c_i - \sum_{j \in \tilde{F}(i) \setminus \{i\}} (x_j - c_j)}{\tilde{O}_i(x) + c_i} \qquad (4.6) \\ &= w(\tilde{F}(i)) \frac{x_i}{\tilde{O}_i(x) + c_i} \geq 0, \end{aligned}$$

where the last inequality follows from (4.5) and (4.4), and because Proposition 3.1 and $x \in C(c_{\mathcal{G}})$ imply $x \geq 0$.

Finally we show that $x = \delta^{\beta}(\mathcal{G})$. For any $i \in N$ it follows from (4.6) that

$$x_i = w_i \frac{\tilde{O}_i(x) + c_i}{w(\tilde{F}(i))}. \tag{4.7}$$

For any k such that $1 \leq k \leq p$, and $i \in N(G^k)$, define the number

$$t_j^k := \begin{cases} 0 & \text{if } i = r_k, \\ \dfrac{\tilde{O}_i(x) + c_i}{w(\tilde{F}(i))} & \text{else.} \end{cases} \tag{4.8}$$

From this definition follows, for any $i \in N(G^k)$, that $t_i^k = \sum_{j \in N(\tilde{P}_i)} (t_j^k - t_{\pi(j)}^k)$. Also, by (4.7) we have $x_i = w_i t_i^k$ for all $i \in N(G^k)$. We will complete the proof by showing that $t_j^k - t_{\pi(j)}^k = \frac{c_j}{w(\tilde{F}(j))}$ for all $j \in N(G^k)$, and by referring to the definition given in (4.1). If

$\pi(j) = r_{k(j)}$, then $\tilde{O}_j(x) = 0$, so the result follows from (4.8). Else

$$\begin{aligned} t_j^k - t_{\pi(j)}^k &= \frac{\tilde{O}_j(x) + c_j}{w(\tilde{F}(j))} - \frac{\tilde{O}_{\pi(j)}(x) + c_{\pi(j)}}{w(\tilde{F}(\pi(j)))} \\ &= \frac{\tilde{O}_j(x) + c_j - \tilde{O}_j(x)}{w(\tilde{F}(j))} = \frac{c_j}{w(\tilde{F}(j))}, \end{aligned}$$

where the second equality follows from (4.2). □

Theorem 4.6 *The set of all down-home allocations equals the core of a maintenance game, i.e.* $\{\delta^\beta(\mathcal{G}) \mid \beta \in \mathcal{B}(\mathcal{G})\} = C(c_{\mathcal{G}})$.

Proof. That the weighted down-home allocations form a superset of $C(c_{\mathcal{G}})$ follows from Proposition 4.5. To show the inclusion, suppose $\beta \in \mathcal{B}(\mathcal{G})$. The proof is complete by first noting that $\delta^\beta(\mathcal{G}^k) \in C(c_{\mathcal{G}^k})$ for every $k = 1, \ldots, p$ by (iv) in Proposition 3.1, and from Proposition 3.6(i). □

In Monderer *et al.* (1992), it is shown in a non-constructive way that the set of weighted Shapley values equals the core for convex games. In order to define the weighted Shapley value of a game, we need to make the following definition.

Definition 4.7 Call an S-weight system for the set of players N an ordered pair $\mu := (\mathcal{S}, \lambda)$, where $\mathcal{S} = (S_1, \ldots, S_q)$ is an ordered partition of the player set N, and $\lambda^{S_\ell} \in \mathbb{R}_{++}^{S_\ell} \cap \Delta(S_\ell))$ for all $\ell = 1, \ldots, q$. Let $\mathcal{M}(N)$ be the set of all S-weight systems for N.

Let $\mu = ((S_1, \ldots, S_q), \lambda)$ be a weight system in $\mathcal{M}(\mathcal{N})$. Define for each $S \subseteq N$, $m(S) := \min\{j : S_j \cap S \neq \emptyset\}$, and let $\bar{S} := S \cap S_{m(S)}$. Then the *weighted Shapley value* corresponding to μ is determined as the linear operator $\Phi^\mu : \Gamma^N \to \mathbb{R}^N$ such that for all $S \subseteq N$,

$$\Phi_i^\mu(u_S) = \begin{cases} \dfrac{\lambda_i}{\lambda(\bar{S})} & \text{if } i \in \bar{S}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

In the unanimity game u_S, the importance of the players depend on how they are "ranked", i.e. where they are located in the ordered collection $\mathcal{S}$. In a cost game the most important players, i.e. those in $\bar{S}$, carry the entire cost. In the case of our cost game $c_{\mathcal{G}}$, because of (2.2), we only need to consider the (dual) unanimity games corresponding to users of arcs, i.e. the games $u^*_{N(B_e)}$ for all $e \in E$. If, for some $e \in E$ and $i \in N$, we have $i \in N(B_e)$, we say that i is *a user of* e. For some $e \in E$, let

$$S(e) := N(B_e) \cap S_{\min\{j: N(B_e) \cap S_j \neq \emptyset\}},$$

and if $i \in S(e)$, we say that i is a *senior user of* e. If i is a user, but not a senior one, of e, there must exist some $j \neq i$ such that $j \in S(e)$, and we say that *i is dominated by j at* e. The weighted Shapley value for a maintenance game is given by the value of the dual unanimity game for each arc,

$$\Phi_i^{\mu}(u^*_{N(B_e)}) = \begin{cases} \dfrac{\lambda_i}{\lambda(S(e))} & \text{if } i \in S(e), \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Example 4.8 Consider the example illustrated in Figure 4.5, where $c(e) := 10$ for all $e \in E$. Let $\mathcal{S} := (\{2,3\}, \{1,4,5\})$ and $\lambda := \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)^T$, hence $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(N)$. The corresponding weighted Shapley value is $\Phi^{\mu}(c_{\mathcal{G}}) = (0, 15, 15, 10, 10)^T$. Player 1 pays nothing, since he is not among the senior users of any arc. Players 2 and 3 dominate all other players at arc $(r, 1)$, and since they both have the same weight, they both pay 5 here. Only player 2 uses $(1, 2)$, so he pays for this arc alone. Since he does not use any other arc except $(r, 1)$, his total contribution is $5 + 10 = 15$. Player 3 dominates all other players at $(1, 3)$, and since he is not using any other arc except $(r, 1)$, his total contribution is $5 + 10 = 15$. Players 4 and 5 are dominated by other players at all arcs that they use, except at the arcs e_4 and e_5, respectively, where they make up the entire set of senior users, and therefore they contribute 10 each. $\triangleleft$

Theorem 4.9

(i) For any $\beta \in \mathcal{B}(\mathcal{G})$, there exists $\mu \in \mathcal{M}(N)$ such that $\Phi^{\mu}(c_{\mathcal{G}}) = \delta^{\beta}(\mathcal{G})$.

(ii) For any $\mu \in \mathcal{M}(N)$, there exists $\beta \in \mathcal{B}(\mathcal{G})$ such that $\Phi^{\mu}(c_{\mathcal{G}}) = \delta^{\beta}(\mathcal{G})$.

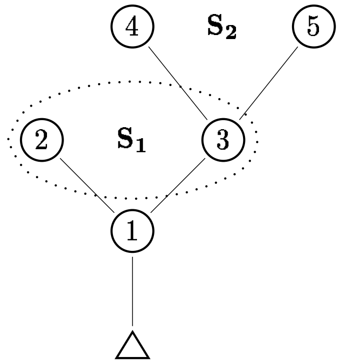

Figure 4.5: Tree for Example 4.8

Proof. (*i*) Let $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ for some maintenance problem $\mathcal{G}$. Note that the elements of $\mathcal{T} = (G^1, \ldots, G^p)$ can be ordered arbitrarily without affecting $\delta^\beta(\mathcal{G})$, and we choose an ordering such that $k(i) < k(j) \Rightarrow j \notin N(P_i)$ for any pair $i, j \in N$. Let, for every $k = 1, \ldots, p$, $S_k := \{i \in N(G^k) \mid w_i > 0\}$ and $S_{p+k} := \{i \in N(G^k) \mid w_i = 0\}$. The ordered collection $(S_1, \ldots, S_p, S_{p+1}, \ldots, S_{2p})$ contains q nonempty elements, where $p \leq q \leq 2p$, and let $\mathcal{S} := (S_1, \ldots, S_q)$ be the ordered collection obtained by deleting the empty elements. Also, for every $i \in N$, let

$$\lambda_i := \begin{cases} \dfrac{w_i}{w(S_{\ell(i)})} & \text{if } w_i > 0, \\ \dfrac{1}{|S_{\ell(i)}|} & \text{otherwise,} \end{cases} \tag{4.11}$$

where $\ell(i) = \ell$ if and only if $i \in S_\ell$. It is easily seen that $\mu := (\mathcal{S}, \lambda) \in \mathcal{M}(N)$. For any $i \in N$, we have

$$\begin{aligned} \Phi_i^\mu(c_\mathcal{G}) &= \sum_{\substack{e \in E \\ S(e) \ni i}} \frac{\lambda_i}{\lambda(S(e))} c(e) = \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} \frac{\lambda_i}{\lambda(S(e))} c(e) \\ &= \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} \frac{w_i}{w(S(e))} c(e) = \sum_{e \in E(\tilde{P}_i)} \frac{w_i}{w(N(\tilde{B}_e))} c(e) = \delta_i^\beta(\mathcal{G}). \end{aligned}$$

The first equality follows from (2.2), the additivity of the weighted Shapley value, and (4.10). The second equality follows from the fact that we can have $i \in S(e)$ only if $e \in E(\tilde{P}_i)$. Suppose, on the contrary, that $i \in S(e)$ for some $e \in E \setminus E(\tilde{P}_i)$. Since we can have $i \in S(e)$ only if i is a user of e, we must have $e \in E(P_i)$. Then, by the construction of $\mathcal{S}$, we must have $N(B_e) \cap S_j \neq \emptyset$ for some $j < \ell(i)$, implying $i \notin S(e)$, a contradiction. In order to prove the third equality, it is sufficient to show that if $i \in S(e)$ for some $i \in N$ and $e \in E(\tilde{P}_i)$ such that $c(e) > 0$, then $\lambda_i = \frac{w_i}{w(S_{\ell(i)})}$, and hence $\lambda(S(e)) = \frac{w(S(e))}{w(S_{\ell(i)})}$. Suppose that this is not true. Then $w_i = 0$ by (4.11), and $\beta \in \mathcal{B}(\mathcal{G})$ implies that there exists some $j \in N(\tilde{B}_e)$ such that $w_j > 0$. Then, by the construction of $\mathcal{S}$, $i \notin S(e)$, a contradiction. The fourth equality follows because, for any $e \in E$ and $i \in N$, $i \in N(\tilde{B}_e) \setminus S(e)$ implies $w_i = 0$ (by the construction of $\mathcal{S}$), so $w(N(\tilde{B}_e)) = w(S(e))$. The last equality follows from (4.1).

(ii) Let $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(N)$ for some maintenance problem $\mathcal{G}$. We construct $\mathcal{T}$ by applying algorithm 4.10.

Algorithm 4.10

Initialization

Let $S'_m := S_m$ for every $m = 1, \ldots, q$, $w := \lambda$, and $\ell := 1$.

Main step

Repeat

    For $i \in S'_\ell$ **do**

        For $j \in N(P_i)$ **do**

            If $\ell(j) > \ell(i)$ **then**

              $S'_{\ell(i)} := S'_{\ell(i)} \cup \{j\}$

              $S'_{\ell(j)} := S'_{\ell(j)} \setminus \{j\}$

              $w_j := 0$

    $\ell := \ell + 1$

until $\ell > q$

The algorithm will give as output the ordered set of coalitions $S'_1, \ldots, S'_q$. Suppose that this ordered set has q' nonempty members. Delete the empty members, and for every $1 \leq \ell \leq q'$, let $G^\ell_1, \ldots, G^\ell_{i_\ell}$ be the collection of pseudo subtrees corresponding to maximal connected, with respect to G, components of S'_ℓ. Clearly, the ordered set

$$G^1_1, \ldots, G^1_{i_1}, G^2_1, \ldots, G^2_{i_2}, \ldots, G^{q'}_1, \ldots, G^{q'}_{i_{q'}}$$

is a partition of G into pseudo subtrees. Let p be the number of members of this partition, re-index, and set $\mathcal{T} := (G^1, \ldots, G^p)$. Since $\lambda \geq 0$, we have $w \geq 0$. Then it follows that $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ since, for any $i \in N$, $w_i = 0$ implies, by algorithm 4.10, that there exists some $j \in \tilde{F}(i) \setminus \{i\}$ such that $w_j > 0$. Now, for every $i \in N$,

$$\delta_i^\beta(\mathcal{G}) = \sum_{e \in E(\tilde{P}_i)} c(e) \frac{w_i}{w(N(\tilde{B}_e))} = \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))}$$
$$= \sum_{\substack{e \in E(P_i) \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))} = \sum_{\substack{e \in E \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))} = \phi_i^\mu(c_{\mathcal{G}}).$$

The first equality follows from (4.1), and the second equality follows from the fact that $w_i = 0$ if $i \notin S(e)$ for some $e \in E(\tilde{P}_i)$, and since $w(N(\tilde{B}_e)) = \lambda(S(e))$ for every $e \in E$. To see that the latter equality is correct, consider some $e \in E$. After applying algorithm 4.10, the vertices in $N(P_j) \cap N(B_e)$ will be included in $S'_{\ell(j)}$ for every $j \in S(e)$. Hence the vertex set $\cup_{j \in S(e)}(N(P_j) \cap N(B_e))$ will be connected, with respect to G, and we must therefore have $S(e) \subseteq N(\tilde{B}_e)$. Also, $j \in N(\tilde{B}_e) \setminus S(e)$ implies $w_j = 0$, and $j \in S(e)$ implies $w_j = \lambda_j$, hence we obtain the desired result. The third equality follows because $e \in E(P_i) \setminus E(\tilde{P}_i)$ implies, from algorithm 4.10, that $N(B_e) \cap S_j \neq \emptyset$ for some $j < \ell(i)$, i.e. i is dominated by the members of S_j ($i \notin S(e)$). The fourth equality follows because $e \in E \setminus E(P_i)$ implies that i is not a user of e, hence $i \notin S(e)$, and the last equality follows from (2.2), the additivity of the weighted Shapley value, and (4.10). □

Example 4.11 Consider the maintenance problem in example 4.2, and the weight system $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$, where $\mathcal{T} = (G^1, G^2)$ and $w = (1, 1, 3, 1)^T$. Here, the corresponding $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(N)$ is uniquely given by $\mathcal{S} := (\{1, 2, 3\}, \{4\})$ and $\lambda = \left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\right)^T$. ◁

Example 4.12 Consider the maintenance game in example 4.8, and the weight system $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(N)$, where $\mathcal{S} = (\{2, 3\}, \{1, 4, 5\})$ and $\lambda = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)^T$. By applying algorithm 4.10, we obtain the partition $\mathcal{S}' = (\{1, 2, 3\}, \{4, 5\})$ of the player set, and the weight vector $w = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)^T$. Note that player 1 has been absorbed by the partition member containing 2 and 3, since 1 is dominated by these two players, and that, accordingly, his weight is now zero. By taking maximal connected subsets of each partition member, we obtain a partition of G into pseudo subtrees, equal to $\mathcal{T} = (G^1, G^2, G^3)$, where $N(G^1) = \{1, 2, 3\}$, $N(G^2) = \{4\}$, and $N(G^3) = \{5\}$. $\triangleleft$

Theorems 4.6 and 4.9 together imply:

Corollary 4.13 *The core of the maintenance game $(N, c_\mathcal{G})$ equals the set of weighted Shapley values, i.e.* $\{\Phi^\mu(c_\mathcal{G}) \mid \mu \in \mathcal{M}(N)\} = C(c_\mathcal{G})$.

Monderer *et al.* (1992) show a more general result, that the set of all weighted Shapley values equals the core of any concave cost game. However, in proving this they needed a fixed point theorem.

5 The core and the set of weighted neighbour-home allocations

In the case of the weighted down-home allocation, the players have an obligation to help their neighbours (predecessors), since they are required to start working from the community center towards their own home. A less extreme social obligation results by applying rules (i)-(iv) in section 4, as well as (v) and (vi) below. The resulting allocation will be called the *neighbour-home* allocation.

(v) If, for any worker $i \in N$, the road between $r_{k(i)}$ and $\pi(i)$ has not been finished yet, then i is working outside his own arc e_i.

(vi) Each worker paints as close to his home as the rules (i)-(v) permit him to.

The algorithm in Maschler *et al.* (1995) returns a special case of the weighted neighbour-home allocation, the nucleolus, where $\mathcal{T} = \{G\}$ and $w_i = \frac{1}{|N|}$ for all $i \in N$. We will show, analogous to the treatment in Section 4 for the weighted down-home allocation, that the set of weighted neighbour-home allocations equals the core, when the weight systems vary over the set $\mathcal{B}(\mathcal{G})$. In order to do this, we need to present the scheme implied by rules (i)-

(v) in a more formal manner, and this is done in Algorithm 5.1. Let $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$, where $\mathcal{T} = (G^1, \ldots, G^p)$. The neighbour-home allocation, denoted $\eta^\beta(\mathcal{G})$, is obtained by, for each of the restricted maintenance problems $\mathcal{G}^k$, $1 \leq k \leq p$, applying 5.1 to the restricted maintenance problem $\mathcal{G}^k$.

Let $x(e,q) \in [0, c(e)]$ be the part of the cost of arc $e \in E(G^k)$ which is paid before stage q. Let $E_q \subseteq E(G^k)$ be the subset of arcs whose cost is covered at stage q, and let $E(q) := \cup_{j<q} E_j$. Let $e(i,q)$ be the arc to which player i is contributing in stage q, and let $S(e,q) := \{i \in N(G^k) \mid e(i,q) = e\}$ be the set of players contributing to arc e in stage q. Let $Q(i)$ denote the first stage in which i stops contributing.

Algorithm 5.1

STEP 0

$q := 1$

$x(e,1) := 0$ for all $e \in E(G^k)$

$E(1) := \emptyset$

$e(i,1) := \begin{cases} e_{\pi(i)} & \text{if } \pi(i) \neq r_k \\ e_i & \text{otherwise} \end{cases}$

STEP 1

For any $e \in E(G^k) \setminus E(q)$ such that $S(e,q) \neq \emptyset$, it would take

$$t(e,q) := \frac{c(e) - x(e,q)}{w(S(e,q))}$$

units of time to finish paying for arc e. Thus, the first arc will be finished after $t(q) := \min\{t(e,q) \mid e \in E(G^k) \setminus E(q) \text{ and } S(e,q) \neq \emptyset\}$ units of time. Then $w(S(e,q))t(q)$ is the fraction of an arc $e \in E(G^k) \setminus E(q)$ which is constructed at stage q, and therefore $x(e, q+1) := x(e,q) + w(S(e,q))t(q)$. Let $E_q := \{e \in E(G^k) \setminus E(q) \mid t(e,q) = t(q)\}$ be the subset of arcs finished at stage q, and let $E(q+1) := E(q) \cup E_q$ be the subset of arcs finished at or before stage $q+1$. Consider every $i \in S(e,q)$ for every $e \in E_q$. If there exists an unfinished arc between $e = e(i,q)$ and the root, i.e. $f \preceq e$ such that $f \in E(G^k) \setminus E(q+1)$, then choose such an f as close to e as possible, and set $e(i, q+1) := f$. If such an arc does not exist, and if i's own arc is not finished, i.e. $e_i \in E(G^k) \setminus E(q+1)$, set $e(i, q+1) := e_i$. Otherwise, set $Q(i) := q$ and $\eta_i^\beta(\mathcal{G}) := \sum_{q=1}^{Q(i)} t(q) w_i$.

STEP 2

If $E(q+1) = E(G^k)$, terminate. Otherwise, set $q := q+1$, and repeat step 1.

We will first demonstrate the algorithm by an example.

Example 5.2 For example 4.2, where $\beta = ((G^1, G^2), (1,1,3,1))$, $N(G^1) = \{1,2,3\}$, and $N(G^2) = \{4\}$, we have $\eta^\beta(\mathcal{G}) = (4, 4, 22, 10)^T$. Player 4 is alone in his pseudo subtree G^2, so he will contribute the entire cost of the arc $(1,4)$, i.e. 10. For pseudo subtree G^1 we apply algorithm 5.1. Initially, $e(1,1) = e(2,1) = (r,1)$ and $e(3,1) = (1,2)$. The first arc is finished after $t(1) = \min\{\frac{10}{2}, \frac{10}{3}\} = \frac{10}{3} = t((1,2),1)$ units of time, and the set of arcs finished in the first stage is $E_1 = \{(1,2)\}$. Now $e(3,2) = (r,1)$ and $S((r,1),2) = \{1,2,3\}$, i.e. all three players will be contributing to arc $(r,1)$ in the second stage. Then $t(2) = t((r,1),2) = \frac{10 - \frac{10}{3}\cdot(1+1)}{5} = \frac{2}{3}$, and $E_2 = \{(r,1)\}$. Players 1 and 2 stop contributing after the second stage, i.e. $Q(1) = Q(2) = 2$, and they each contribute, in total, $1 \cdot \left(\frac{10}{3} + \frac{2}{3}\right) = 4$. Player 3 now starts contributing to his own arc, i.e. $e(3,3) = (2,3)$. He will finish this arc in $t(3) = \frac{10}{3}$ units of time, and then stop contributing ($Q(3) = 3$). His total contribution is $3 \cdot \left(\frac{10}{3} + \frac{2}{3} + \frac{10}{3}\right) = 22$. Since all the arcs have been finished after stage 3, the algorithm terminates. $\triangleleft$

Now we turn to the following question: Given a core element y, can we find a weight system $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $y = \eta^\beta(\mathcal{G})$? It turns out that the answer is yes, and proposition 3.6(ii) suggests that we choose $\mathcal{T} := \mathcal{T}(x)$. We will now illustrate, using two examples, how the weight vector w can be found.

Example 5.3 Consider Example 4.4 again, where $y = (4, 12, 12, 12)^T$ is a core element, for which the partition $\mathcal{T}(y)$ is trivial. First, note that arc e_1 will be finished after $\frac{4}{w_1}$ units of time. Moreover, players 2 and 3 will be contributing at this arc until it is finished, and will return home (to finish their own arcs) exactly when this is the case. In order to calculate how long 2 and 3 will be contributing at arc e_1, we need to find their *far-away* *contributions*, i.e. how much they contribute at arcs other than their own. We will do this by first finding their *home contributions*, i.e. how much they contribute at their own arcs. Player 2's home contribution is obviously given by the cost of his own arc, i.e. 10, since he has no followers other than himself. Thus his far-away contribution, i.e. the amount that he will contribute at arc e_1, is 12 - 10 = 2. For player 3 the picture is more complicated, since he has a follower, player 4. Rule (vi) implies that if 4 contributes anything above the cost of his own arc, this contribution will first be used to finish arc e_3. This is indeed the case, since player 4 contributes 12 - 10 = 2 in excess of the cost of his own arc. The home contribution of player 3 will thus be only 10 - 2 = 8. Since he contributes 12 in total, his far-away contribution will be 12 - 8 = 4. To sum up, we know now that players 1, 2, and 3 contributes 4, 2, and 4, respectively, at arc e_1. This implies

$$\frac{4}{w_1} = \frac{2}{w_2} = \frac{4}{w_3}. \tag{5.1}$$

Player 4 will be contributing at e_3 until this arc is finished. His contribution at this arc is 2, as we stated above. Player 3 will stop contributing at all exactly when his own arc is finished, at which point he will have contributed 12. This implies

$$\frac{12}{w_3} = \frac{2}{w_4}. \tag{5.2}$$

A weight vector that satisfies both (5.1) and (5.2) is $w := (4, 2, 4, \frac{4}{6})^T$. $\triangleleft$

To formalize the notions of home and far-away contributions, let $y = \eta^\beta(\mathcal{G})$ for some weight system $\beta \in \mathcal{B}(\mathcal{G})$, and let $i \in N(G^k)$, $1 \leq k \leq p$. Note that the contribution of the players in $\tilde{F}(i) \setminus \{i\}$ at or below e_i will be given by what they contribute in excess of the cost of their own arcs, i.e. by $\sum_{j \in \tilde{F}(i) \setminus \{i\}} (y_j - c_j)$. Because of rule (vi), this excess contribution will first be used at arc e_i. Player i will cover the remaining part $c_i - \sum_{j \in \tilde{F}(i) \setminus \{i\}} (y_j - c_j)$ of the cost of his own arc, if this expression is positive. Hence, player i's home contribution is given by

$$h_i(y) := \left(c_i - \sum_{j \in \tilde{F}(i) \setminus \{i\}} (y_j - c_j) \right)_+ = \left(c_i - \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \tilde{O}_j(y) \right)_+ ,$$

and his far-away contribution is $f_i(y) := y_i - h_i(y)$. Next we will consider an example where some players contribute nothing, which makes finding the weight vector slightly more complicated.

Example 5.4 Consider the maintenance problem illustrated in Figure 5.1, and the corresponding core element $y = (0, 12, 16, 0, 16, 16)^T$. We set the weights of players that does not make any contribution, to zero, i.e. $w_1 := w_4 := 0$.

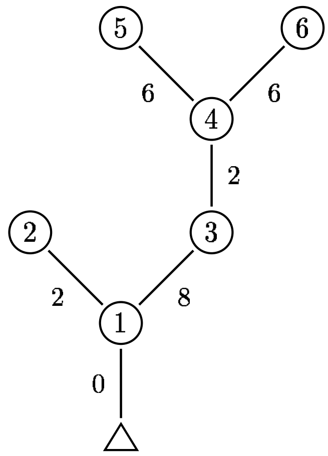

Figure 5.1: The tree corresponding to Example 5.4.

Player 2 has no followers other than himself, he will have to contribute to the entire cost of his own arc e_2, i.e. $h_2(y) = 10$. The remaining 12 - 10 = 2 (= $f_2(y)$) that he contributes, will be towards the cost of arc e_1. Player 3 will also contribute at this arc, but how much? The answer can be found by noting that the followers of 3 (except himself), i.e. players 4, 5 and 6, contribute 0 + 16 + 16 = 32, while the total cost of their own arcs is only 30. Hence we have $h_3(y) = 10 - (32 - 30) = 8$ and $f_3(y) = 16 - 8 = 8$. Players 2 and 3 will return home at exactly the same time, i.e. when arc e_1 is finished. This will happen after

$$\frac{f_2(y)}{w_2} = \frac{2}{w_2} = \frac{f_3(y)}{w_3} = \frac{8}{w_3} \tag{5.3}$$

units of time. Note that the weights of players 2 and 3 are not related to the weight of the player in front of them, as was the case in the Example 5.3.

We have $h_5(y) = h_6(y) = 10$, since neither player 5 nor player 6 have followers other than themselves, and therefore $f_5(y) = f_6(y) = 16 - 10 = 6$. Because of rule (v), they cannot return home until the players in front of them have all finished. The last such player to finish will be the closest one that makes a positive contribution, i.e. player 3, who finishes after $\frac{16}{w_3}$ units of time. Our weight vector must therefore satisfy

$$\frac{6}{w_5} = \frac{6}{w_6} = \frac{16}{w_3}. \tag{5.4}$$

A weight vector that assigns weight zero to players that does not contribute anything, as well as satisfies (5.3) and (5.4), is given by $w = (0, 4, 16, 0, 6, 6)^T$. $\triangleleft$

For any $i \in N(G^k)$, let $\pi^+(i)$ be the first predecessor of i in G^k such that $y_i > 0$. If no such predecessor exists, let $\pi^+(i) := r_k$. Also, let $N^+(G^k) := \{i \in N(G^k) \mid y_i > 0\}$. Note that if $i \in N(G^k)$ is such that $\pi(i) \neq r_k$ and $\tilde{O}_i(y) > 0$, then he will contribute a nonzero amount to the cost of the arcs in $E(\tilde{P}_{\pi^+(i)})$, and will return home exactly when all the arcs in $E(\tilde{P}_i) \setminus \{e_i\}$ have been finished. Since $\frac{f_i(y)}{w_i}$ is the total time that player i spends contributing to arcs other than his own, we have

$$\pi^+(i) = \pi^+(j) \neq r_k \Rightarrow \frac{f_i(y)}{w_i} = \frac{f_j(y)}{w_j} \qquad \text{for all } i, j \in N^+(G^k). \tag{5.5}$$

Also, if a player contributes to the cost of the arcs of his predecessors, he will return home exactly when the last one of his predecessors stops contributing, i.e.

$$\frac{f_i(y)}{w_i} = \frac{y_{\pi^+(i)}}{w_{\pi^+(i)}} \qquad \text{for all } i \in N^+(G^k) \text{ such that } \pi^+(i) \neq r_k. \tag{5.6}$$

Let, for $k = 1, \ldots, p$, $B^k(x) := \{i \in N(G^k) : \pi^+(i) = r_k\}$.

Proposition 5.5 *Let $x \in C(c_{\mathcal{G}})$. There exists $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $x = \eta^\beta(\mathcal{G})$, where $\mathcal{T} = \mathcal{T}(x)$, and w satisfies, for every $k = 1, \ldots, p$,*

$$\frac{f_i(x)}{w_i} = \frac{f_j(x)}{w_j} \qquad \forall i, j \in N^+(G^k) \cap B^k(x), \tag{5.7}$$

$$\frac{f_i(x)}{w_i} = \frac{x_{\pi^+(i)}}{w_{\pi^+(i)}} \qquad \forall i \in N^+(G^k) \setminus B^k(x), \tag{5.8}$$

$$w_i = 0 \qquad \forall i \in N(G^k) \setminus N^+(G^k). \tag{5.9}$$

Proof. Existence: Clearly, $\mathcal{T}(x)$ exists. Let $1 \leq k \leq p$. In order to show that (5.7)-(5.9) has a solution, note that, for every $i \in N(G^k)$,

$$x_i > 0 \Rightarrow f_i(x) = x_i - \left(c_i - \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \tilde{O}_j(x) \right)_+ > 0. \tag{5.10}$$

Hence a solution can be found by arbitrarily fixing $w_i > 0$ for some $i \in N^+(G^k) \cap B^k(x)$. Then, for every $j \in N^+(G^k) \cap B^k(x) \setminus \{i\}$, w_j is given by (5.7). For each $j \in N^+(G^k) \cap B^k(x)$ and $\ell \in \tilde{F}(j)$, w_ℓ is given by applying (5.8) in a recursive manner.

Let $y := \eta^\beta(\mathcal{G})$. Then

$$x_i > 0 \Leftrightarrow w_i > 0 \Leftrightarrow y_i > 0 \qquad\qquad \forall i \in N. \tag{5.11}$$

Let $i \in N(G^k)$ for $1 \leq k \leq p$. $x_i > 0 \Leftrightarrow w_i > 0$ follows from (5.10) and the construction of w described above. $y_i > 0 \Rightarrow w_i > 0$ follows from algorithm 5.1. Note that, since $w_i > 0 \Rightarrow x_i > 0$, and from $x \in C(c_\mathcal{G})$ and Proposition 3.1(iv), there must exist some arc $e \in E(\tilde{P}_i)$ such that $c(e) > 0$. Then, since y has been constructed using algorithm 5.1, we have $w_i > 0 \Rightarrow y_i > 0$. Because of (5.11), we have $B^k(x) = B^k(y)$, and the definitions of $\pi^+(\ )$ and $N^+(G^k)$ are unambiguous.

We claim that $\beta \in \mathcal{B}(\mathcal{G})$. Clearly, $\mathcal{T} = \mathcal{T}(x)$ is a partition of G into pseudo subtrees. $x \geq 0$ (from Proposition 3.1), together with (5.9) and (5.11), imply $w \geq 0$. Let $1 \leq k \leq p$, and let $c_i > 0$ for some $i \in N(G^k)$. Since $x^{N(G^k)} \in C(c_{\mathcal{G}^k})$ by Proposition 3.6(ii), we must have $\tilde{O}_i(x) = \sum_{j \in \tilde{F}(i)}(x_j - c_j) \geq 0$ by Proposition 3.1(iii). Since $x_j \geq 0$ and $c_j \geq 0$ for all $j \in \tilde{F}(i)$, there must exist some $\ell \in \tilde{F}(i)$ such that $x_\ell > 0$, and $w_\ell > 0$ then follows from (5.11).

We claim that $x = \eta^\beta(\mathcal{G})$: Let $1 \leq k \leq p$. We have

$$y(N(G^k)) = c_{\mathcal{G}^k}(N(G^k)) = x(N(G^k)), \tag{5.12}$$

where the first equality follows from Algorithm 5.1 and $\beta \in \mathcal{B}(\mathcal{G})$, and the second from $x \in C(c_\mathcal{G})$ and Proposition 3.6(ii). If $N(G^k) = \{i\}$ for some $i \in N$, then $x_i = y_i$ follows directly, so we will assume in the following that $|N(G^k)| > 1$. Suppose, contrary to our claim, that $x^{N(G^k)} \neq y^{N(G^k)}$. By (5.12), there must exist some $i, j \in N(G^k)$ such that $i \neq j$, $x_i < y_i$, and $x_j > y_j$. Note that w satisfies (5.8) with respect to x, by definition, and with respect to y, by (5.6). Therefore we must have $f_\ell(x) < f_\ell(y)$ for every $\ell \in (\pi^+)^{-1}(i)$, so there must exist some $m \in \tilde{F}(\ell)$ such that $x_m < y_m$. By repeating this argument with $i := m$, we can show that, for every leaf ℓ of G^k such that $\ell \in \tilde{F}(i)$, we have $x_\ell < y_\ell$. Similarly, for every leaf ℓ in G^k such that $\ell \in \tilde{F}(j)$, we have $x_\ell > y_\ell$. So $\tilde{F}(i) \cap \tilde{F}(j) = \emptyset$, otherwise we would have a contradiction. Since $\tilde{F}(i) \cap \tilde{F}(j) = \emptyset$ is true for *any* choice of $i, j \in N^+(G^k)$ such that $x_i < y_i$ and $x_j > y_j$, we must have $x_\ell \leq y_\ell$ for all $\ell \in \tilde{F}(i)$, and $x_\ell \geq y_\ell$ for all $\ell \in \tilde{F}(j)$. Therefore, $f_\ell(x) \leq f_\ell(y)$ for all $\ell \in \tilde{F}(i)$ and $f_\ell(x) \geq f_\ell(y)$ for all $\ell \in \tilde{F}(j)$. Now, since (5.8) implies $x_\ell = \frac{f_m(x)}{w_m} w_\ell$ and $y_\ell = \frac{f_m(y)}{w_m} w_\ell$ for every $\ell \in N^+(G^k)$ and $m \in (\pi^+)^{-1}(\ell)$, we must have $i, j \in B^k(x)$, which contradicts (5.7). $\square$

In the same way that Proposition 4.5 enabled us to prove Theorem 4.6, Proposition 5.5 enables us to prove that the set of neighbour-home allocations equals the core.

Theorem 5.6 *For any maintenance problem $\mathcal{G}$ the set of all neighbour-home allocations equals $C(c_\mathcal{G})$, i.e. $\{\eta^\beta(\mathcal{G}) \mid \beta \in \mathcal{B}(\mathcal{G})\} = C(c_\mathcal{G})$.*

We know, from Maschler *et al.* (1995), that $\eta^{(\mathcal{T},w)}(\mathcal{G})$ is equal to the nucleolus of $c_\mathcal{G}$ if we set $\mathcal{T} = \{G\}$ and $w_i = \frac{1}{|N|}$ for all $i \in N$. In Yanovskaya (1992) , the *weighted nucleolus* is defined by replacing the ordinary excess function by a weighted excess function, and it is shown that every point in the relative interior of the core can be obtained as a weighted nucleolus. For the game (N, g), and some pre-imputation x, this weighted excess function is given by, for any $S \neq N, \emptyset$, $\epsilon^p(S, x) := p_S(g(S) - x(S))$, where $p_S > 0$. Let $\beta := ((G^1, \ldots, G^p), w) \in \mathcal{B}(\mathcal{G})$ for some maintenance problem $\mathcal{G}$, and let $k = 1, \ldots, p$. Suppose we set $p_S := f(w^{N(G^k)})$ for all $S \subset N(G^k)$ such that $S \neq \emptyset$, where $f : \mathbb{R}^{N(G^k)} \rightarrow \mathbb{R}$. An interesting open problem is whether we can pick the function f such that $\eta^\beta(\mathcal{G})$, when restricted to the members of $N(G^k)$, is the weighted nucleolus of the game $c_{\mathcal{G}^k}$.

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